Yuya Tanizaki

1. Introduction to basics of QFT

(cf. Polyakov, Gauge Fields and Strings; Witten, Dynamical Aspects of QFT)

QFT

Microscopic world: Theory of everything, String, Lattice, etc. $\xrightarrow{\text{RG flow}}$ Macroscopic pheno.: Effective QFT

Examples:

Ising model	$\longrightarrow$	ϕ^4 theory
Spin chain	$\longrightarrow$	σ model
QCD	$\longrightarrow$	pion effective theory
Electroweak	$\longrightarrow$	Four-fermi theory
String	$\longrightarrow$	Various gauge theories

1-1 Formulation of QFT

Let us restrict our attention to relativistic QFT.
$\parallel$
Quantum Mechanics & Special Relativity

We investigate the Euclidean formulation.

example (ϕ^4 theory)

$$S[\phi] = \int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right\}.$$

Correlation functions (or Schwinger functions)

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z}\int \mathcal{D}\phi \, e^{-S[\phi]} \phi(x_1) \cdots \phi(x_n).$$

Lorentz invariance $\Leftrightarrow$ Euclidean rotational invariance

$$\langle \phi(O\cdot x_1) \cdots \phi(O\cdot x_n)\rangle = \langle \phi(x_1) \cdots \phi(x_n)\rangle \quad \text{w/ } O \in SO(d).$$

Unitarity $\Leftrightarrow$ Reflection positivity

$$\langle \overline{F[\phi](x)} \cdot F[\phi](0) \rangle \geq 0.$$

Osterwalder-Schrader reconstruction theorem (cf. Glimm, Jaffe; Quantum Physics)

Given the set of Schwinger functions satisfying

- Euclidean inv.
- Reflection positivity
- some analyticity conditions,

then we can construct $\mathcal{H}$ and $\hat{H}$ giving relativistic QFT.

(Quiz: Check this reconstruct for harmonic oscillator)

1-2. Global symmetry

Here, I want to clarify the definitions of symmetry, spontaneous symmetry breaking, and give a brief summary of various theorems.

Definition (Symmetry) (cf. Gaiotto, Kapustin, Seiberg, Willet)

d-dim. QFT is said to have a global symmetry G if the following holds

X : d-dim. spacetime (Riemannian manifold)

$U_g(M_{d-1})$: an operator defined on $M_{d-1} \subset X$ (($d-1$)-dim. sub-manifold), which is labelled by $g \in G$.

- (Group law) $U_{g_1}(M_{d-1})\, U_{g_2}(M_{d-1}) = U_{g_1 g_2}(M_{d-1})$.
- (Conservation law) $U_g(M_{d-1})$ is topological, i.e.
$$\langle U_g(M_{d-1} + \underbrace{\Delta M_{d-1}}_{\text{small deformation}})\, \mathcal{O}(x_1)\cdots\rangle = \langle U_g(M_{d-1})\, \mathcal{O}(x_1)\cdots\rangle .$$
- (Transformation) Any local operators should be able to be written with $V_i(x)$, which satisfies
$$U_g(\underbrace{S_0^{d-1}}_{(d-1)\text{-sphere around } x=0})\, V_i(0) = \underbrace{R(g)_i{}^j}_{\text{representati: } R \text{ of } G} V_j(0).$$
- For some $V_i(0) \neq 0$, R should be a $\underbrace{\text{faithful representation}}_{\text{i.e. } \forall g \neq e \quad R(g) \neq 1}$ of G. //

· Continuous symmetry

G : a continuous group

$S[\phi]$ is inv. under G.

$\delta_\varepsilon \phi$ is an infinitesimal change with $\varepsilon \in \mathfrak{g} = \mathrm{Lie}(G)$, then

$$\int \mathcal{D}\phi \left(e^{-S[\phi + \delta_\varepsilon \phi]} - e^{-S[\phi]} \right)$$
$$= -\int \mathcal{D}\phi\, e^{-S[\phi]} \Big(\int_x \underbrace{\delta_\varepsilon \phi \frac{\delta S}{\delta \phi}}_{\substack{\parallel \\ d\varepsilon^a \wedge j^a}} \Big)$$

so that $=0$ for constant ε.

This j^a defines the Noether current.

We define $U_g(M_{d-1})$ for infinitesimal change as

$$U_{e^{\varepsilon}}(M_{d-1}) = \exp\left(\varepsilon^a \int_{M_{d-1}} j^a\right).$$

This operator is topological since

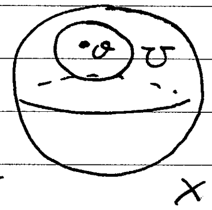
$$d j^a = 0.$$

Spontaneous symmetry breaking

On compact spacetime X,

$$\langle \mathcal{O}(x) \rangle = 0$$

if $\mathcal{O}$ is in a nontrivial rep. of some symmetry G.



((Proof)

$$R(g) \cdot \langle \mathcal{O}(x) \rangle = \langle U_g(S^{d-1}_x)\, \mathcal{O}(x) \rangle = \langle \mathcal{O}(x) \rangle$$

and $R(g) \neq 1$ for some g. $\Longrightarrow$ $\langle \mathcal{O}(x) \rangle = 0$.)

Def. (SSB)

We say that the symmetry G is spontaneously broken, if $\exists\, \mathcal{O}(x)$: a local op. in a nontrivial rep. G s.t.

$$\langle \mathcal{O}(x)^* \mathcal{O}(0) \rangle \underset{\substack{|x| \to \infty \\ \mathrm{vol}(X) \to \infty}}{\not\to} 0.$$

//

(That is, cluster decomposition does not hold.)

$\rightarrow$ By taking a vacuum state which satisfy the cluster decomp, $\langle \mathcal{O}(x) \rangle \neq 0$.

example

Let us consider the Ising model $\hat{H} = -\sum_{n=1}^{L} \sigma_z(n) \cdot \sigma_z(n+1)$, and put the periodic boundary conditi $\sigma_z(n+L) = \sigma_z(n)$.

Two ground states:

$$|+\rangle = |\uparrow\uparrow\cdots\uparrow\rangle, \qquad |-\rangle = |\downarrow\downarrow\cdots\downarrow\rangle.$$

$$\langle \sigma_z \rangle \simeq \langle + | \sigma_z | + \rangle + \langle - | \sigma_z | - \rangle = 1 + (-1) = 0.$$

$$\langle \sigma_z(x) \cdot \sigma_z(0) \rangle \simeq \langle + | \sigma_z(x) \cdot \sigma_z(0) | + \rangle + \langle - | \sigma_z(x) \cdot \sigma_z(0) | - \rangle = 1^2 + (-1)^2 = 2.$$

Thus, $\mathbb{Z}_2$ symmetry $\sigma_z \mapsto -\sigma_z$ is spontaneously broken. //

No SSB in quantum mechanics?

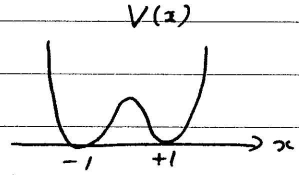
There's a folklore saying that there's no SSB in quantum mechanics.

- What is the theorem for this folklore?
- Is it always true?

Motivat:

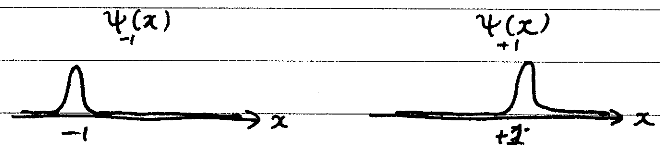
Consider QM with double-well potential:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}(x^2-1)^2.$$



System has a $\mathbb{Z}_2$ symmetry: $x \mapsto -x$.

Classical minima:



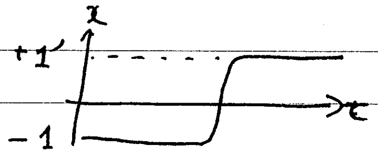
Quantum ground state: $\psi(x) \simeq \psi_{-1}(x) + \psi_{+1}(x)$.

In Euclidean path integral,

$$Z = \int \mathcal{D}x\, e^{-\int_0^\beta d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(x^2-1)^2\right)}$$

There is a contributi from instanton:

$$x_I(-\infty) = -1, \quad x_I(+\infty) = +1$$



(and also anti-instanton).

Peforming dilute instanton gas approximati (DIGA), we find

$$\langle x(\tau)\, x(0)\rangle \sim \exp(-C\, e^{-S_I}|\tau|).$$

$$\to 0 \quad \text{as } \tau \to \infty.$$

Thus, SSB does not occur.

cf. $$\left.\begin{aligned} |GS\rangle &\sim |+\rangle + |-\rangle \\ |1^{st}\rangle &\sim |+\rangle - |-\rangle \end{aligned}\right\} \Delta E \sim e^{-S_I}.$$

(Quiz: Perform DIGA to derive the two-point correlator)

From this example, we learn that, even when classical vacua break symmetries, quantum ground state restores it by instantons.
More precisely,

Th'm $\mathcal{H} = L^2(\mathbb{R})$, $\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x)$ with nonsingular confining $V(x)$.
Then,

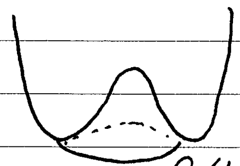
- ground state is unique,
- the ground-state wavefunction can be taken to be positive for any x.

(Proof) We consider $e^{-\hat{H}}$. Its path-integral rep. in coord. basis is given by

$$\underbrace{\langle x_f | e^{-H} | x_i \rangle}_{\gneq 0} = \int_{x(0)=x_i}^{x(1)=x_f} \mathcal{D}x \ \underbrace{e^{-\int d\tau \{\frac{1}{2}\dot{x}^2 + V(x)\}}}_{>0}$$

Then, apply Perron-Frobenius theorem. //

(Quiz: Consider examples that does not have unique ground state)



G/H

SSB of continuous symmetry

When continuous symmetry is spontaneously broken, there exist NG massless boson for each broken generator (Nambu-Goldstone theorem).

example Consider complex scalar theory,

$$S = \int d^dx \{ \partial_\mu \phi^* \partial_\mu \phi + V(\phi^* \phi) \}.$$

$V(x) = (x^2 - v^2)^2$ is the wine-bottle potential.

Theory has the $U(1)$ symmetry, $\phi \mapsto e^{i\alpha}\phi$.

We parametrize $\phi \simeq v e^{i\pi}$, then

$$S \simeq v^2 \int d^dx \, (\partial_\mu \pi)^2.$$

$\pi(x)$ gives the massless boson field. In this parametrization, the broken $U(1)$ generator is $j_\mu = i\phi^* \overleftrightarrow{\partial_\mu} \phi \simeq 2v\partial_\mu \pi$, so the broken generator creates NG boson. //

Coleman - Mermin - Wagner thm

In 2d, continuous symmetry cannot be spontaneously broken.

(Proof)

This is a corollary of Nambu-Goldstone thm.
Assume for contradiction that a continuous sym. G is spontaneously broken, then a broken generator $j_\mu \sim \partial_\mu \pi$ creats massless boson. However,

$$\langle \pi(x)\,\pi(0)\rangle = \int \frac{d^2k}{(2\pi)^2}\,\frac{e^{ik\cdot x}}{k^2} = -\ln\frac{|x|}{R} < 0,$$

and thus $\pi(x)$ breaks reflection positivity. This is the contradiction. //

1-3 Gauge redundancy

Assume that we have a d-dim. QFT with symmetry G. Then, we can try to construct another QFT without symmetry G, and this procedure is called gauging symmetry.

The idea is the following. We consider the partition functi with the source field A for the current j:

$$Z(A) = \int \mathcal{D}\phi\, e^{-S[\phi] + i\int A\wedge j + \mathcal{O}(A^2)}$$

Since $dj = 0$ by conservati law, we can (at least naively) expect that

$$Z(A + \underbrace{d\varepsilon}_{\text{gauge parameter}} + \mathcal{O}(A\varepsilon)) = Z(A).$$

This says that A is the connection of a principal G-bundle $P \to X$, so we can write

$$Z(P) = \int \mathcal{D}\phi\, e^{-S[\phi] + i\int A\wedge j + \mathcal{O}(A^2)}.$$

If this is true, then we can define another partition functi

$$Z' = \int \mathcal{D}a\, e^{-\frac{1}{g^2}\int F\wedge * F}\, Z(a),$$

where $\int \mathcal{D}a$ is the sum over all possible G-bdls and connections on it, F is the curvature of the G-bdl.

(Note: The above statements make sense also for discrete symmetries)

Thm (Elitzur)

Gauge-dependent local operator is a null operator.

(Proof).

Let $\mathcal{O}(x)$ be a gauge-dependent operator. Since we are summing over all possible gauge fields,

$$\langle \mathcal{O}(x) \cdots \rangle = \langle U_g(S_x^{d-1})\, \mathcal{O}(x) \cdots \rangle$$
$$= R(g) \cdot \langle \mathcal{O}(x) \cdots \rangle .$$

$\Rightarrow \langle \mathcal{O}(x) \cdots \rangle = 0$ unless $R(g) = 1$ for any $g \in G$. //

Because of this, gauge invariance is not symmetry in our definition.

Applicati: to phenomenological Lagrangian

Let's assume that a d-dim. QFT has symmetry G, and $G \xrightarrow{SSB} H$, where H is a subgroup of G.

Since the broken generators define a coordinate of G/H, the low-energy Lagrangian is the G/H sigma model.

To construct this, we first consider the principal G model

$$\mathcal{L} = \int d^d x \; \mathrm{tr}\left[\, |g^\dagger d g|^2 \right] + \cdots .$$

($g^\dagger d g$ is the generator of G, and called Maurer–Cartan form).

In this case, G is completely broken.

We want to eliminate massless bosons belonging to $\mathfrak{h} = \mathrm{Lie}(H)$.

This can be done by gauging H:

$$\mathcal{L}_{\text{gauged}} = \int d^d x \; \mathrm{tr}\left[\, \underbrace{|g^\dagger D_A g|}_{}{}^2 \right] .$$

$$\alpha_\perp = g^\dagger d g \cdot (1 - P_{\mathfrak{h}}) .$$ //

(Quiz: Check the consistency with the standard textbook derivati:)

2. Anomaly, Anomaly Matching

Yuya Tanizaki

When we explain the gauging procedure, we assume that the partition function $Z(A)$ is well-defined as a function of principal G-bundles. Possibly, there exist obstructions for this assumption, which is called <u>anomaly</u>.

2-1 't Hooft anomaly

<u>Def.</u> ('t Hooft anomaly)

d-dim. QFT with symmetry G has an 't Hooft anomaly, if the following occurs:

Let X : d-dim. Riem. mfd.

A : G-gauge field on X

$A \mapsto A + d_A\theta$: G-gauge transformation,

then the partition function $Z[A]$ is not G-gauge invariant:

$$Z[A + d_A\theta] = Z[A]\, e^{i\int_X \mathcal{A}(\theta, A)}$$

where

$$\mathcal{A} \neq \delta_\theta \text{(any } d\text{-dim. local functional of } A\text{)}. \quad //$$

One of the most famous example would be chiral anomaly. So, let us briefly review this, and check that it matches the above definition.

<u>example</u> (Chiral anomaly).

$d = 2n$-dimensional left-handed fermion:

$$\mathcal{L} = \sum_{f=1}^{N} \bar{\Psi}_f \gamma^\mu \partial_\mu P_L \Psi_f .$$

Ψ_f : N-flavor Dirac fermions. γ^μ : gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$.

$\gamma = -i^n \gamma^1\gamma^2 \cdots \gamma^{2n}$ (so that $(\gamma^\mu)^\dagger = \gamma^\mu$, $\gamma^\dagger = \gamma$, $\gamma^2 = 1$)

$P_{L/R} = \frac{1 \mp \gamma}{2}$: chiral projector.

We introduce a background gauge field A by minimal coupling

$$\mathcal{L} = \cancel{\sum_f}\ \cancel{\bar{\Psi}_f}\ \bar{\Psi} \gamma^\mu \underbrace{(\partial_\mu + A_\mu)}_{D_\mu} P_L \Psi .$$

We want to diagonalize $\gamma^\mu D_\mu P_L$, but there is a problem.

Since $\gamma^\mu P_L = P_R \gamma^\mu$,

$$\gamma^\mu D_\mu P_L : (\text{left-handed spinors}) \longrightarrow (\text{right-handed spinors}).$$

different spaces

Therefore, the eigenvalue problem of $\gamma^\mu D_\mu P_L$ is ill-defined.

To define the eigenvalue problem, we add the free right-handed fermions that don't couple to A: (cf. Alvarez-Gaume, Ginsparg)

$$\mathcal{L} = \bar\psi \gamma^\mu (\partial_\mu + A_\mu P_L)\psi$$
$$= \bar\psi \gamma^\mu (\partial_\mu P_R + D_\mu P_L)\psi.$$

Eigenvalue problem:
$$\begin{cases} (\not\partial P_R + \not D P_L)\phi_n = \lambda_n \phi_n \\ \bar\phi_n (\overleftarrow{\not\partial} P_R + \overleftarrow{\not D} P_L) = \lambda_n \bar\phi_n. \end{cases}$$

We assume that completeness relation holds
$$\int d^dx\ \bar\phi_n \phi_m = \delta_{nm}.$$

We now expand the Dirac field ψ and $\bar\psi$ as
$$\begin{cases} \psi(x) = \sum_n a_n \phi_n(x) \\ \bar\psi(x) = \sum_n \bar\phi_n(x)\ b_n \end{cases}$$

where a_n, b_n are fermionic variables. The Lagrangian becomes
$$\int d^dx\ \mathcal{L} = \sum_n b_n \lambda_n a_n.$$

We define the regularized path-integral measure by
$$\mathcal{D}\bar\psi\,\mathcal{D}\psi = \prod_n e^{\lambda_n^2/\Lambda^2}\, db_n\, da_n.$$

Gauge transformation: ω: infinitesimal gauge parameter
$$\begin{cases} A \mapsto A' = A + [D, \omega], \\ \psi \mapsto \psi' = \psi - \omega P_L \psi, \\ \bar\psi \mapsto \bar\psi' = \bar\psi + \bar\psi P_R \omega. \end{cases}$$

The Lagrangian is invariant, and the mode decomposition is changed as
$$a'_n = \int d^dx\ \bar\phi_n \psi' = \sum_m \left(\delta_{nm} - \int d^dx\ \omega(x)\ \bar\phi_n(x) P_L \phi_m(x)\right) a_m$$
$$b'_n = \int d^dx\ \bar\psi' \phi_n = \sum_m b_m \left(\delta_{mn} + \int d^dx\ \omega(x)\ \bar\phi_m(x) P_R \phi_n(x)\right)$$

$\Rightarrow$

anomaly.

$$\mathcal{D}\bar\psi\,\mathcal{D}\psi = \mathcal{D}\bar\psi'\,\mathcal{D}\psi'\, e^{i\mathcal{A}}.$$

(Fujikawa)

We can compute $\mathcal{A}$ as

$$\mathcal{A} = \sum_n e^{\lambda_n^2/\Lambda^2}\left(-\int d^dx\, \bar{\phi}_n\, \omega P_L \phi_n + \int d^dx\, \bar{\phi}_n\, \omega P_R \phi_n\right)$$

$$= \mathrm{Tr}\left[-\omega P_L \exp\left(\frac{1}{\Lambda^2}(\not{\partial} P_R + \not{D} P_L)^2\right) + \omega P_R \exp\left(\frac{1}{\Lambda^2}(\not{\partial} P_R + \not{D} P_L)^2\right)\right].$$

We can simplify this by algebra that

$$\mathcal{A} = \frac{1}{2}\mathrm{Tr}\left[\omega\gamma\left(e^{\not{\partial}\not{D}/\Lambda^2} + e^{\not{D}\not{\partial}/\Lambda^2}\right)\right]$$

$$\underbrace{-\frac{1}{2}\mathrm{Tr}\left[\omega\left(e^{\not{\partial}\not{D}/\Lambda^2} - e^{\not{D}\not{\partial}/\Lambda^2}\right)\right]}.$$

(Quiz: Show that 2^{nd} term is fake as anomaly, i.e. it is δ_θ (d-dim local terms).).

By accepting the result of quiz, the genuine anomaly is

$$\mathcal{A} = \frac{1}{2}\mathrm{Tr}\left[\omega\gamma\left(e^{\not{\partial}\not{D}/\Lambda^2} + e^{\not{D}\not{\partial}/\Lambda^2}\right)\right].$$

This is what is called non-Abelian consistent anomaly.

(Quiz: Evaluate $\mathcal{A}$ for $d = 2, 4$. For example, when $d=4$, you should get

$$\mathcal{A} = -\frac{1}{24\pi^2}\int \mathrm{tr}\left[\omega\, d\left(AdA + \frac{1}{2}A^3\right)\right].$$) //

Anomaly-inflow (Callan-Harvey)

If the anomaly $\mathcal{A}$ satisfies

$$\int_X \mathcal{A} = \int_{M_{d+1}} \delta_\theta\, S_{d+1}[A],$$

where $S_{d+1}[A]$ is $(d+1)$-dimensional topological action with $\partial M_{d+1} = X$, then we say that anomaly-inflow condition is satisfied.

Assumption

In this lecture, we assume that any 't Hooft anomaly satisfies anomaly-inflow assumption.

<u>example</u> (Stora-Zumino chain for chiral anomaly).

We review Stora-Zumino chain for chiral anomaly to see that chiral anomaly satisfies the anomaly-inflow assumption.

We assign the fermionic statistics to the gauge parameter ω, (i.e. ω is a Faddeev-Popov ghost) (i.e. ω is a Grassmannian variable). Then, δ_ω defined by

$$\begin{cases} \delta_\omega A = [D, \omega], \\ \delta_\omega \omega = \omega \wedge \omega, \end{cases}$$

is a BRST differential, i.e. $\delta_\omega^2 = 0$.
We also require $d\delta_\omega + \delta_\omega d = 0$.
(Quiz: Check the nilpotency of δ_ω).

Now, the chiral anomaly $\mathcal{A}$ is given by

$$\int_X \mathcal{A} = \delta_\omega \ln Z[A],$$

where $Z[A]$ is the partition func. of chiral fermions coupled to A.

Nilpotency of δ_ω requires that

$$\delta_\omega \int_X \mathcal{A} = \delta_\omega^2 \ln Z = 0,$$

and this condition for anomaly $\mathcal{A}$ is called the Wess-Zumino consistency condition.

$\Rightarrow$ Anomaly $\mathcal{A}$ is a d-dim. local functional, that satisfies
- BRST closed condition (WZ consistency),
- with ghost number 1.

Stora-Zumino chain is an ad-hoc way to construct such $\mathcal{A}$:
Assume that we have $(d+2)$-dim. gauge-inv. functional $P_{d+2}(A)$:

$$\begin{cases} \delta_\omega P_{d+2}(A) = 0 \\ d\, P_{d+2}(A) = 0. \end{cases}$$

(Specifically, we take $(d=2n)$
$$P_{d+2} = \frac{1}{(2\pi)^{n}(n+1)!} \mathrm{tr}[F^{n+1}].$$
)

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 & & & \\
\Omega^2_{2n} \arrow[r, "d"] & \delta\Omega^1_{2n} \arrow[u, "\delta"] \arrow[r, "d"] & 0 & & \\
 & \boxed{\Omega^1_{2n}} \arrow[u, "\delta"] \arrow[r, "d"] & \delta\Omega^0_{2n+1} \arrow[u, "\delta"] \arrow[r, "d"] & 0 & \\
 & & \Omega^0_{2n+1} \arrow[u, "\delta"] \arrow[r, "d"] & P_{2n+2}(A) \arrow[u, "\delta"'] \arrow[r, "d"] & 0
\end{tikzcd}
$$

Stora - Zumino chain.

Now, let us regard $\Omega^1_{2n} = \mathcal{A}$, then

- $\mathcal{A}$ has the ghost number 1

and

$$
\begin{aligned}
\delta \int_X \mathcal{A} &= \int_X \delta \Omega^1_{2n} \\
&= \int_X d\,\Omega^2_{2n} \\
&= 0
\end{aligned}
$$

so

- $\int_X \mathcal{A}$ is BRST closed.

Therefore, if chiral anomaly is given by

$$Z[A + \delta_\omega A] = Z[A] \exp\left(\int_X \Omega^1_{2n}\right),$$

then, preparing $(d+1)$-dim. manifold M_{d+1} s.t. $\partial M_{d+1} = X$,

$$Z[A]\, e^{-\int_{M_{d+1}} \Omega^0_{2n+1}[A]}$$

is gauge invariant by anomaly-inflow.

(Quiz: Check explicitly that

- chiral anomaly satisfies Stora-Zumino chain with $P_{2n+2} = \frac{1}{(2\pi)^n (n+1)!} tr(F^n)$,
- and Ω^1_{2n+1} is $(2n+1)$-dim. level-1 Chern-Simons action.)

"$CS_{2n+1}[A]$"

2-2. Anomaly matching condition

We've seen that anomaly is an obstruction to define the theory on principal G-bundle. This has an important physical consequence on IR behavior of G-symmetric QFT. Such condition is called the 't Hooft anomaly matching condition ('t Hooft; Frishman, Schwimmer, Banks, Yankielowicz; ....).

Thm (Anomaly matching).

Consider a d-dim. QFT with symmetry G, and assume that it has an 't Hooft anomaly,

$$Z[A+\delta_\theta A] = Z[A] \exp\left(i\int_x \mathcal{A}(\theta, A)\right).$$

Then, the IR effective field theories must reproduce the same anomaly:

$$Z_{IR}[A+\delta_\theta A] = Z_{IR}[A] \exp\left(i\int_x \mathcal{A}(\theta, A)\right).$$

//

Note. The theorem may sound trivial, but imply a very nontrivial result.

As we've seen in chiral anomaly, the anomaly $\mathcal{A}$ is a local functional of θ and A. This is true in general, meaning that the obstruction comes from the UV properties of the QFT. (Indeed, one can translate the chiral anomaly to the Schwinger term of current-current OPEs.)

Since the fields in IR effective theories are smeared, one might expect that such short-range properties of original QFT disappears. The above theorem says the anomaly is exception to this naive expectation, and IR physics must reproduce the same anomaly. (So, one cannot take genuine classical limits if anomaly exists.)

Because of this thm, people often says that 't Hooft anomaly is RG invariant.

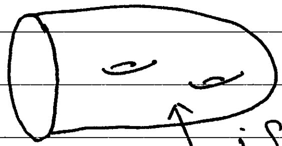
//

(Proof of anomaly matching).

The meaning of RG flow is that IR properties of those two QFTs are the same: As $\mathrm{vol}(X) \to \infty$,

$$Z(A) - Z_{IR}(A) \to 0.$$

Using the anomaly-inflow assumption, one can regard our QFT is a boundary of $(d+1)$-dim. classical TQFT $S_{d+1}[A]$ (this is called symmetry-protected topological order):



our QFT $\to Z[A]$ $\qquad e^{iS_{d+1}[A]}$: $(d+1)$-dim. SPT protected by G.

so that

$$Z[A]\, e^{iS_{d+1}[A]}$$

is gauge invariant.

Since $S_{d+1}[A]$ is topological, it does not change by $\mathrm{vol}(X) \to \infty$.

Thus

$$0 = \delta_\theta \ln\left[Z[A]\, e^{iS_{d+1}[A]}\right]$$
$$\xrightarrow{\mathrm{vol}(X)\to\infty} \delta_\theta \ln\left[Z_{IR}[A]\, e^{iS_{d+1}[A]}\right] = 0.$$

Therefore,

$$\delta_\theta \ln\left[Z_{IR}[A]\right] = \delta_\theta \ln\left[Z_{UV}[A]\right] = i\int_X \mathcal{A}(\theta, A). \quad //$$

Short summary:

When anomaly exists, we should regard our QFT as a boundary of G-protected SPT when we introduce A.

Then, IR theory of the boundary should cancel the gauge anomaly of anomaly inflow.

Especially, when 't Hooft anomaly exists, then the system cannot have the unique and gapped ground state. Instead, it must have

- conformal behavior
- spontaneous symmetry breaking
- topological order

and their combinations.

Examples of anomaly matching

Here, we discuss some examples of anomaly matching. (The most classic example by 't Hooft will be discussed later (Sec. 4), so we here consider simpler examples.)

(example 1) Quantum mechanics on S^1; a particle of single spin.

We consider a QM with a single spin,

$$\hat{H} = J \hat{S}_z^2 .$$

The Hilbert space is spanned by $(2S+1)$ states: $\mathcal{H} = \sum_{M=-S}^{S} |S, M\rangle$,

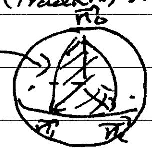
$$\begin{cases} \vec{S}^2 |S, M\rangle = S(S+1) |S, M\rangle \\ S_z |S, M\rangle = M |S, M\rangle . \end{cases}$$

To derive the path integral, we first define the coherent state:

$$|\vec{n}\rangle = \exp(i\,\theta\,(\vec{n}_0 \times \vec{n}) \cdot \vec{S}\,)\, |S, S\rangle$$

with $\cos\theta = (\vec{n}_0 \cdot \vec{n})$, and $\vec{n}_0$ is a unit vector along z-direction. (Wiegmann; Fradkin, Stone)

This satisfies the following: solid angle among $\vec{n}_1, \vec{n}_2, \vec{n}_0$.



$$\begin{cases} \langle \vec{n}_1 | \vec{n}_2 \rangle = e^{iS\,\Phi(\vec{n}_1, \vec{n}_2, \vec{n}_0)} \left(\frac{1+\vec{n}_1\cdot\vec{n}_2}{2}\right)^S , \\ \langle \vec{n} | \vec{S} | \vec{n} \rangle = S\,\vec{n} , \\ \mathbb{1} = \int \left(\frac{2S+1}{4\pi}\right) d^3\vec{n}\, \delta(\vec{n}^2 - 1) \cdot |\vec{n}\rangle\langle\vec{n}| . \end{cases}$$

(Quiz: Check these relations for coherent ~~states~~ states).

Using this, the Euclidean effective action for small imaginary-time development

$$\langle \vec{n}(t_f) | e^{-\delta\epsilon H} | \vec{n}(t_i) \rangle$$

$$= \exp\left[\, i S\, \Phi(\vec{n}_f, \vec{n}_i, \vec{n}_0) + S \ln\left(\frac{1+\vec{n}_f \cdot \vec{n}_i}{2}\right) - \delta\epsilon \langle \vec{n}_f | H | \vec{n}_i \rangle \right]$$

$\vec{n}_i$ up to $\mathcal{O}(\delta\epsilon)$

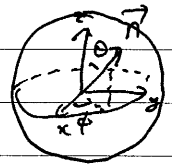
Thus, the Euclidean action is

$$S_E = \underbrace{iS \int \vec{n} \cdot (d\vec{n} \wedge d\vec{n})}_{\text{Wess-Zumino term.}} - S\delta\epsilon \int d\epsilon\, (\partial_\epsilon \vec{n})^2 - J \int d\epsilon\, n_z^2$$

$$\parallel$$

$$\int (1-\cos\theta)\, d\phi$$

Meaning of WZ term:



$$\int \mathcal{D}\phi\, e^{iS\int(1-\cos\theta)\,d\phi} \Rightarrow \delta(\, S(1-\cos\theta) \in \mathbb{Z}\,).$$

$\Leftrightarrow -1 \leq \cos\theta = \frac{-n+S}{S} \leq 1 \rightarrow n = 0, \cdots, 2S$

so we get $(2S+1)$ states.

Now, we consider the case $J \gg 0$.
Then, the low-energy states are given by
$$n_z \simeq 0.$$
This sets $\theta = \frac{\pi}{2}$. As we set $\theta = \frac{\pi}{2}$ in Euclidean action, we get
$$S_E = i\frac{(2\pi S)}{2\pi}\int d\phi - S\delta t\int(\partial_t\phi)^2 dt,$$
and we get a particle on $S^1 \ni \phi$ with the theta term, and
$$\theta = 2\pi S.$$

Let us analyze this system in view of symmetry and anomaly.
When $J=0$, the spin rotation symmetry was $SO(3)$: (Note: Even when $S=\frac{1}{2}$, symmetry is $SO(3)$, not $SU(2)$)
$$\vec{n} \mapsto O\cdot\vec{n} \quad \text{with } O^T\cdot O = 1,\ \det O = 1.$$
Applying $J n_z^2$, this spin symmetry is explicitly broken to
$$O(2) = SO(2) \rtimes \mathbb{Z}_2 \subset SO(3).$$
$$\begin{cases} SO(2): & \phi \mapsto \phi + \alpha \\ \mathbb{Z}_2: & \phi \mapsto -\phi. \end{cases}$$
(we call this the charge conjugation).

$SO(2)$ is trivially a symmetry, but $(\mathbb{Z}_2)_C$ is a little tricky.

$$S_E \xrightarrow{C} -i\frac{(2\pi S)}{2\pi}\int d\phi - S\delta t\int(-\partial_t\phi)^2 dt$$
$$= S_E - \underbrace{2iS\int d\phi}_{\in (2S)2\pi\mathbb{Z}}.$$
Since $S \in \frac{1}{2}\mathbb{Z}$, S_E is C-inv. mod $2\pi i\mathbb{Z}$.
In the path integral, we only consider $\exp(S_E)$,
and thus the system has $\mathbb{Z}_2$ symmetry.

We now gauge the $SO(2) = U(1)$ symmetry, and call the background gauge field as A.

$$Z[A] = \int \mathcal{D}\phi \, \exp\left[-S\delta t \int (\partial_t \phi + A_t)^2 dt + iS\int (d\phi + A)\right].$$

$$\downarrow C$$

$$\begin{aligned} Z[-A] &= \int \mathcal{D}\phi \, \exp\left[-S\delta t \int (\partial_t \phi + A_t)^2 dt - iS\int (d\phi + A)\right] \\ &= \int \mathcal{D}\phi \, \exp\left[-S\delta t \int (\partial_t \phi + A_t)^2 dt + iS\int (d\phi + A)\right] \\ &\quad \times \exp\left[-i \underbrace{(2S)\int d\phi}_{\in 2\pi\mathbb{Z}} - i(2S)\int A\right] \\ &= Z[A] \cdot \exp\left(-i(2S)\int A\right). \end{aligned}$$

(i) $S = 1, 2, \cdots$.

We should notice that $\exp(ik\int A)$ is $U(1)$ gauge inv., if $k \in \mathbb{Z}$. Thus, let us consider

$$Z_k[A] \equiv Z[A] \, e^{ik\int A}.$$

Applying the charge conjugation, we get

$$\begin{aligned} Z_k[A] &\longmapsto \left(Z[A] e^{-i(2S)\int A}\right) \cdot \left(e^{ik\int A} \cdot e^{-2ik\int A}\right) \\ &= Z_k[A] \exp\left(-i(2k + 2S)\int A\right). \end{aligned}$$

Since S is integer, ~~by~~ we can take $k = -S$, so that $Z_k[A]$ is $U(1)$ gauge inv. & C-inv.

(ii) $S = \frac{1}{2}, \frac{3}{2}, \cdots$.

In this case, there are no $k \in \mathbb{Z}$ s.t. $Z_k[A]$ is both $U(1)$ gauge inv & C-inv.

This is $SO(2) \rtimes \mathbb{Z}_2$ anomaly.

$\rightarrow (\mathbb{Z}_2)_C$ is spontaneously broken in order to match the anomaly.

(Quiz: Convince yourself that this anomaly is the consequence of projective rep. of $SO(3)$, i.e. the double cover $SU(2)$ acts faithfully on $\mathcal{H}$).

(Quiz: What's wrong with the uniqueness of ground state?)

By introducing 2d TQFT, $Z[A] \exp\left(-iS\int_{M_2} dA\right)$, the system can be inv. under both $U(1)$ gauge tran. & C.

This is the anomaly inflow in this example. //

3. 2d nonlinear sigma model

Yuya Tanizaki

3-1. Haldane conjecture

In this section, we study the long-range behavior of 1d anti-ferromagnetic spin chain:

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \qquad (J>0)$$

Since $J>0$, the classical ground states should look like

That is, at the short range, the system would show Neel order.

Interestingly, Haldane noticed that the low-energy theory is described by relativistic QFT! (Haldane, '83)

(Note: This is already nontrivial. For example, ferromagnetic spin chain is non-relativistic, since magnon (NG boson) shows $E = \# k^2$ (quadratic dispersion)
(Proof). In ferro., $\langle S_z \rangle \neq 0$. Since $[S_x, S_y] = S_z$, the commutator of two conserved currents have non-zero VEV. In this case, system has 1 type-II NG mode. //
(Check Hidaka; Watanabe, Murayama...))

After some computations, we obtain the S^2 sigma model with the theta term:

$$S_E = \frac{1}{2g} \int |d\vec{m}|^2 + i \frac{\theta}{2\pi} \frac{1}{4} \int \vec{m} \cdot (d\vec{m} \wedge d\vec{m}).$$

with $\vec{m}^2 = 1$, and

$$g \sim \frac{1}{S}, \qquad \theta = 2\pi S.$$

(Derivation). By applying the path-integral rep. for each spin, we get

$$S_E = \int dt \sum_i J S^2 \vec{n}_i(x_0) \cdot \vec{n}_{i+1}(x_0) + iS \sum_j \int_{t=0}^{1} \int_{t=0}^{\beta} \vec{n}_j \cdot (d\vec{n}_j \wedge d\vec{n}_j)$$

We decompose $\vec{n}_i$ into the slow mode $\vec{m}_i$ and the fast mode $\vec{l}_i$. Since the short-range physics is the Neel order, we add the staggering phase to slow variables

$$\vec{n}_i = (-1)^i \vec{m}_i + \vec{l}_i.$$

We require $\vec{m}_i^2 = 1$, $\vec{m}_i \cdot \vec{l}_i = 0$.

Substitute this and expand in terms of $\vec{l}$ up to the quadratic order. Integrating out $\vec{l}$, we get the result. //

(Quiz: Complete the above derivation).

I prefer the $\mathbb{C}P^1$ description, so let's introduce the spinon field $z \in \mathbb{C}^2$ with $z^\dagger \cdot z = 1$:

$$\vec{m}(x) = z^\dagger(x) \vec{\sigma}\, z(x).$$

Since this decomposition introduces the local $U(1)$ invariance,

$$z(x) \mapsto z(x)\, e^{i\alpha(x)}, \quad z^\dagger(x) \mapsto z^\dagger(x)\, e^{-i\alpha(x)},$$

we should remove it by $U(1)$ gauge field a. The result is

$$S_E = \frac{1}{2g} \int |(d + ia) z|^2 + i \frac{\theta}{2\pi} \int da.$$

(Quiz: Check the classical equivalence between these descriptions).

Now, let us consider the effect of the magnitude of spin S. It appears as the coupling constant,

$$g \sim \frac{1}{S}.$$

However, in QFT, the coupling runs under RG flow, so the classical value of g should not affect the low-energy physics.

What is more important is the relation to θ:

$$\theta = 2\pi S.$$

Since the theta term is topological, this classical value is meaningful even quantum mechanically.

What we mean "topological" here is that

$$\frac{1}{2\pi} \int da = \frac{1}{4} \int \vec{m} \cdot (d\vec{m} \times d\vec{m}) \in \mathbb{Z},$$

which defines the winding number $\pi_2(S^2) = \mathbb{Z}$.

We are only interested in $\exp(-S_E)$ in quantum mechanics, thus $\theta \sim \theta + 2\pi$. Therefore,

$$\begin{cases} S = 1, 2, 3, \ldots & \Longrightarrow \theta = 0 \\ S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots & \Longrightarrow \theta = \pi \end{cases}$$

and we get two different field theories depending on whether S is in integers or in half-integers.

By this observation, Haldane conjectured that

$$\begin{cases} S = 1, 2, \cdots \Rightarrow \text{The anti-ferro. spin chain is gapped} \\ \qquad \text{(since } \mathbb{CP}^1 \text{ sigma model is asymptotically free (Polyakov, '75))} \\ S = \frac{1}{2}, \frac{3}{2}, \cdots \Rightarrow \text{— " — is gappless.} \\ \qquad \text{(since spin-}\frac{1}{2}\text{ chain is gappless by Bethe ansatz).} \end{cases}$$

3.2. Anomaly matching for 2d $\mathbb{CP}^1$ model

Here, we analyze symmetry and anomaly of 2d $\mathbb{CP}^1$ sigma model, and prove (half of) Haldane conjecture. This is first done by Affleck, Lieb '85, but the presentation here is based on Komargodski, Sharon, Thorngren, Zhou '17.

Symmetry group of this theory is

$$SO(3)_{spin} \times (\mathbb{Z}_2)_{lattice}.$$

- Spin rotation symmetry $SO(3)\ (= SU(2)/\mathbb{Z}_2)$

The S^2 sigma model is inv. under $SO(3)$ rotation,

$$\vec{m} \mapsto O \cdot \vec{m} \quad \text{with} \quad O \in SO(3).$$

This is evident in S^2 model, but becomes a bit tricky in $\mathbb{CP}^1$ model. Since $\vec{m} = z^\dagger \vec{\sigma} z$, one might wonder if the symmetry is enlarged to $SU(2)$:

$$z \mapsto U \cdot z \quad \text{with} \quad U \in SU(2).$$

Relation between U and O is

$$U^\dagger \sigma^i U = O^i{}_j \sigma^j.$$

However, both $\pm U$ give the same $O \in SO(3)$, since $SU(2)$ is a double cover of $SO(3)$.

(Q) Does the spin symmetry enhance to $SU(2)$ from $SO(3)$?
→ No, this is impossible. We must take into account the $U(1)$ gauge redundancy.

Any gauge-inv. operators, $z^\dagger \vec{\sigma} z$, da, $\cdots$, is inv under $z \mapsto -z$ ($\mathbb{Z}_2 \subset SU(2)$)
Thus, the symmetry group with faithful rep. on physical local operators is $SO(3)$.

- ($\mathbb{Z}_2$) lattice symmetry, emerged from lattice translation.

The path-integral weight $\exp(-S_E)$ is invariant under
$$\vec{m} \mapsto -\vec{m}.$$
(Since $\det\begin{pmatrix}-1 & & \\ & -1 & \\ & & -1\end{pmatrix} = -1$, this is not an element of $SO(3)$).
In the $\mathbb{CP}^1$ description, this corresponds to
$$\begin{cases} z \mapsto i\sigma_2 z^{\dagger}, \\ a \mapsto -a. \end{cases}$$

Then,
$$\begin{aligned} S_E &= \frac{1}{g}\int \cancel{(d+a)}\,((d-ia)z^{\dagger}) \wedge *((d+ia)z) \\ &\quad + i\frac{\theta}{2\pi}\int da \\ &\mapsto \frac{1}{g}\int (d+ia)z^{t}\cdot i\sigma^2 \wedge (i\sigma^2) * ((d-ia)z^{*}) \\ &\quad - i\frac{\theta}{2\pi}\int da \\ &= S_E - i\frac{2\theta}{2\pi}\int da. \end{aligned}$$
Since $\theta = 2\pi S \in \pi\mathbb{Z}$, $\exp(-S_E)$ does not change.

This originates from the lattice translational symmetry, since $\vec{m}$ is related to the spin vector $\vec{n}$ as
$$\vec{m}(x) = (-1)^x\,\vec{n}(x).$$
Thus, the lattice translation by one lattice unit gives
$$\vec{m}(x) \mapsto -\vec{m}(x+a) \simeq -\vec{m}(x).$$
We now understand why we have a specific relation
$$\theta = 2\pi S.$$
In order to remember the lattice translation symmetry, we cannot change θ from discrete values $2\pi S \in \pi\mathbb{Z}$.
(For other θ's, this $\mathbb{Z}_2$ symmetry doesn't exist)

Now, we understand the symmetry of 2d $\mathbb{CP}^1$ model. Let's try to gauge it, and check whether 't Hooft anomaly exists or not.

Gauging SO(3)

Let us introduce a background gauge field for $SO(3)_{spin}$. Since $\mathfrak{su}(2) = \mathrm{Lie}(SU(2)) = \mathrm{Lie}(SO(3)) = \mathfrak{so}(3)$, there is no difference between $SU(2)$ gauge field and $SO(3)$ gauge field at the LOCAL level. However, there is difference globally since $\pi_1(SU(2)) = 0$ but $\pi_1(SO(3)) = \mathbb{Z}_2$ (Some $SO(3)$ principal bundle does not have a lift to $SU(2)$ principal bundle.)

To see this, we first introduce $SU(2)$ (1-form) gauge field:

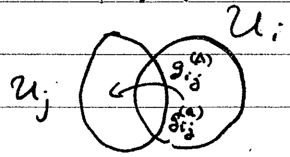
$$A = A^a_\mu \sigma^a dx^\mu .$$

By the minimal coupling procedure, we get

$$S_E = \frac{1}{2g}\int |(d + ia + iA) z|^2 + i\frac{\theta}{2\pi}\int da .$$

Here, we should remind that the gauge fields are 1-forms only locally, (i.e. in a single patch of the 2-manifold X).

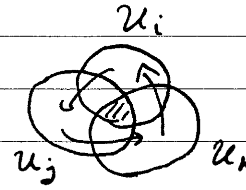
Take an opencover $\{\mathcal{U}_i\}_i$ of X, then one must introduce transition functions on double overlaps $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$, so that



$$\begin{cases} A_j = g^{(A)\dagger}_{ij} A_i\, g^{(A)}_{ij} - i\, g^{(A)\dagger}_{ij} d\, g^{(A)}_{ij}, \\ a_j = a_i - i\, g^{(a)\dagger}_{ij} d\, g^{(a)}_{ij}, \\ z_j = g^{(a)}_{ij} \otimes g^{(A)}_{ij}\, z_i, \end{cases} \qquad -(*)$$

where $g^{(a)}_{ij} : \mathcal{U}_{ij} \to U(1)$ and $g^{(A)}_{ij} : \mathcal{U}_{ij} \to SU(2)$.

Let us discuss the consistency on triple overlaps $\mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$. On triple overlap, by applying transitions $(*)$ three times as shown in figure, one should get the original field by consistency. One obtains



$$\begin{cases} g^{(A)}_{ki}\, g^{(A)}_{jk}\, g^{(A)}_{ij} \in \mathbb{Z}_2 = Z(SU(2)), \\ (g^{(a)}_{ki} \otimes g^{(A)}_{ki})(g^{(a)}_{jk} \otimes g^{(A)}_{jk})(g^{(a)}_{ij} \otimes g^{(A)}_{ij}) = 1 . \end{cases}$$

If we say that we gauge $SU(2)$, we require that

$$g^{(A)}_{ki}\, g^{(A)}_{jk}\, g^{(A)}_{ij} = 1,$$

but this does not have to hold in our case.

More generally, one can introduce $\mathbb{Z}_2$ phase $e^{i\pi n_{ijk}}$ on each U_{ijk}, and require that

$$\begin{cases} g^{(A)}_{ki}\, g^{(A)}_{jk}\, g^{(A)}_{ij} = e^{i\pi n_{ijk}}, \\ g^{(a)}_{ki}\, g^{(a)}_{jk}\, g^{(a)}_{ij} = e^{-i\pi n_{ijk}}. \end{cases}$$

$(n_{ijk} \in \mathbb{Z}_2)$.

We can perform a $\mathbb{Z}_2$ transformation without affecting connection formula (*):

$$\left.\begin{aligned} g^{(A)}_{ij} &\mapsto g^{(A)}_{ij}\, e^{i\pi\lambda_{ij}} \\ g^{(a)}_{ij} &\mapsto g^{(a)}_{ij}\, e^{-i\pi\lambda_{ij}} \end{aligned}\right\} - (**).$$

$(\lambda_{ij} \in \mathbb{Z}_2)$, with

$$n_{ijk} \mapsto n_{ijk} + \lambda_{ij} + \lambda_{jk} + \lambda_{ki}.$$

Consistency of transition functions $g^{(A)}_{ij}, g^{(a)}_{ij}$ on quadruple ~~ope~~ overlap U_{ijkl} gives the condition

$$n_{ijk} - n_{ijl} + n_{ikl} - n_{jkl} = 0 \quad \text{mod } 2.$$

The equivalence class $[\{n_{ijk}\}]$ by taking into account the redundancy (**) is characterized by

$$H^2(X, \mathbb{Z}_2),$$

and this is called 't Hooft magnetic flux.

Only when $[\{n_{ijk}\}] = 0$, the $SO(3)$ bdl has a lift to $SU(2)$ bdl.

A convenient continuum description is given by Kapustin, Seiberg '14.
Here, we introduce the $U(2)$ gauge field $\tilde{A}$ instead of $SU(2)$ gauge field A.

$$S_E = \int |(d + ia + i\tilde{A})z|^2 + i\frac{\theta}{2\pi}\int da$$

We also introduce the $U(1)$ 2-form gauge field B, and require that

$$2B = d(\mathrm{tr}(\tilde{A}))$$

This constraint has the invariance under $U(1)$ 1-form transformations,

$$\begin{cases} B \mapsto B + d\lambda, \\ \tilde{A} \mapsto \tilde{A} + \lambda \mathbb{1}_2, \end{cases}$$

and we postulate this invariance everywhere. Then, $U(2)$ bdl is reduced to $SO(3) = \frac{U(2)}{U(1)}$ bundle.

Invariance of the kinetic term says that

$$a \mapsto a - \lambda.$$

As a consequence, we get

$$S_E = \int |(d + ia + i\tilde{A})z|^2 + i\frac{\theta}{2\pi}\int (da + B).$$

- $\underline{SO(3)\times\mathbb{Z}_2 \text{ anomaly}}$

Since we have gauged $SO(3)$ symmetry, let us check whether we can keep $\mathbb{Z}_2$.

$$
\begin{aligned}
S_E &= \int |(d+ia+i\tilde{A})z|^2 + i\frac{\theta}{2\pi}\int(da+B) \\
&\overset{\mathbb{Z}_2}{\longmapsto} \int |(d+ia+i\tilde{A})z|^2 - i\frac{\theta}{2\pi}\int(da+B) \\
&= \int |(d+ia+i\tilde{A})z|^2 + i\frac{\theta}{2\pi}\int(da+B) \\
&\qquad - i2\cdot\frac{2\pi S}{2\pi}\int(da+B) \\
&= S_E - i(2S)\int(da+B).
\end{aligned}
$$

Since $(2S)\in\mathbb{Z}$ and $\int da\in 2\pi\mathbb{Z}$, we get

$$e^{-S_E} \overset{\mathbb{Z}_2}{\longmapsto} e^{-S_E}\cdot e^{+i(2S)\int B}.$$

Therefore

$$Z[(\tilde{A},B)] \overset{\mathbb{Z}_2}{\longmapsto} Z[(\tilde{A},B)]\cdot e^{i(2S)\int B}$$

We should check whether this additional phase is a genuine or fake anomaly. The possible local counterterm is $ik\int B$ $(k\in\mathbb{Z})$.

So, let us define

$$Z_k = Z\cdot e^{ik\int B},$$

then

$$Z_k \overset{\mathbb{Z}_2}{\longmapsto} Z_k\cdot e^{i(2S+2k)\int B}.$$

- When $S=1,2,\cdots$, we can find the counter term $k=-S$ so that Z_{-S} is $SO(3)\times\mathbb{Z}_2$ inv.
  $\to$ No anomaly.
  $\therefore$ The existence of trivial gap is consistent.

- When $S=\frac{1}{2},\frac{3}{2},\cdots$, no counterterm can cancel the anomaly. One has to prepare 3d SPT to cancel it by anomaly inflow:

$$Z[\tilde{A}\,B]\cdot \exp\left(-S\int_{M_3} dB\right).$$

$\therefore$ $SO(3)\times\mathbb{Z}_2$ has an 't Hooft anomaly.

For $S = \frac{1}{2}, \frac{3}{2}, \cdots$, the possible low-energy behaviors are

$\begin{cases} \bullet \ \text{CFT} \\ \bullet \ \text{SSB of } (\mathbb{Z}_2)_{\text{lattice}}. \end{cases}$

(Coleman-Mermin-Wagner thm prohibits SSB of $SO(3)$.
TQFT cannot appear for 1d spin chain (Wen, '10)).

For CFT scenario, we can show that $SU(2)_k$ WZW matches the anomaly for $k = 1, 3, 5, \cdots$.

$SU(2)$ WZW theories has the chiral symmetry,

$$G_{WZW} = \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$$

and we consider its diagonal subgroup and discrete chiral symmetry

$$G_{WZW} \supset \underbrace{\frac{SU(2)_V}{\mathbb{Z}_2}}_{SO(3)_{\text{spin}}} \times \underbrace{(\mathbb{Z}_2)_L}_{(\mathbb{Z}_2)_{\text{lattice}}}.$$

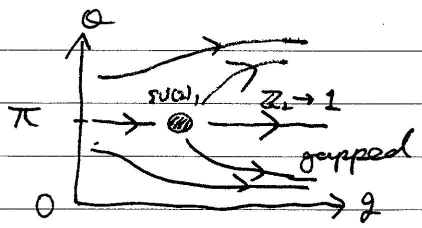
This is first found by Affleck, Haldane '87.

Zamolodchikov's C-theorem implies that CFT with smaller C has a more chance to appear. For $SU(N)_k$ WZW,

$$c = \frac{k(N^2-1)}{k+N},$$

and thus, the most "natural" candidate is $SU(2)_1$ WZW.

RG flow for 2d $\mathbb{CP}^1$



- All these analysis is recently extended to $SU(N)$ spin chain. For details, see Tanizaki, Sulejmanpasic '18.

also Lajko, Wamer, Mila, Affleck '17, Ohmori, Seiberg, Shao '18

4. 4d non-Abelian gauge theories

Yuya Tanizaki

4-1. Pure Yang-Mills theory, Dashen phenomenon at $\theta=\pi$

In this section, we study 4d $SU(N)$ pure Yang-Mills theory especially by paying attention to its θ dependence. We'd like to see it from various aspects, and I am planning to review the results from

- large-N 't Hooft limit (Witten '80)
- softly broken $\mathcal{N}=1$ super Yang-Mills
- anomaly matching (Gaiotto, Kapustin, Komargodski, Seiberg '17)

We first define our theory. Let X be a 4-manifold.
(For simplicity, we always assume that X admits the spin structure).
a : $SU(N)$ gauge field on X.
$G = da + ia^2$: field strength of a.
Then, the Yang-Mills action is defined by
$$S = \frac{1}{2g^2}\int \mathrm{tr}[G\wedge *G] + i\frac{\theta}{8\pi^2}\int \mathrm{tr}[G\wedge G].$$

Point-like observables : $\mathrm{tr}[G_{\mu\nu}G_{\mu\nu}]$, $\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}[G_{\mu\nu}G_{\rho\sigma}]$, ⋯.
Line observable : $W(C) = \mathrm{tr}[\mathcal{P}\exp(i\oint_C a)]$, W^2_C ⋯.

The theory does not have ordinary symmetry except for Poincaré symmetry $SO(4)\ltimes\mathbb{R}^4$ and C, P, T at $\theta = 0, \pi$.

We still define the distinction of phases by the behavior of the Wilson loop: When the system is gapped, then
$$W(C) \xrightarrow{C\to\infty} \begin{cases} e^{-\#\,\mathrm{Area}(C)} & \text{(Confining phase)} \\ e^{-\#'\,\mathrm{Perimeter}(C)} & \text{(Higgs phase)}. \end{cases}$$

In order to include this into Landau's classification of phases, we generalize the notion of symmetry.
This is called higher-form symmetry
(Gaiotto, Kapustin, Seiberg, Willett '14).

Definition (p-form Symmetry)

d-dim. QFT is said to have a p-form symmetry G if the following holds:

X : d-dim. Riem. mfd.

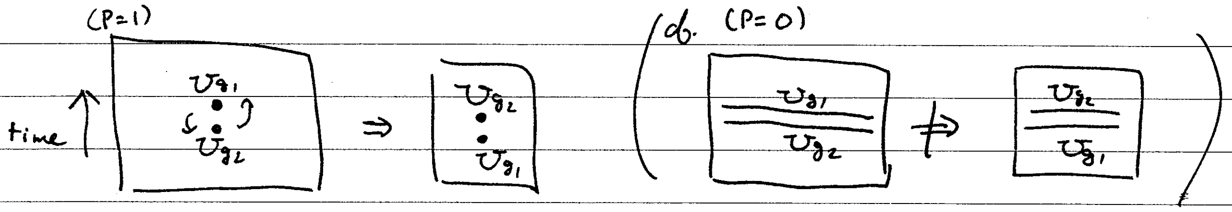
$U_g(M_{d-p-1})$: an operator defined on $M_{d-p-1} \subset X$ ((d-p-1)-dim. submfd) which is labelled by $g \in G$.

- (Group law) $U_{g_1}(M_{d-p-1})\, U_{g_2}(M_{d-p-1}) = U_{g_1 g_2}(M_{d-p-1})$.
- (Conservati law) $U_g(M_{d-p-1})$ is topological.
- (Transformati) For operators $V(C^{(p)})$ defined on p-dim. submanifold $C^{(p)}$, U_g and V has commutati relat
$$U_g(\underbrace{S^{d-p-1}_0}_{\substack{(d-p-1)\text{-sphere} \\ \text{that link with } C^{(p)}}})\, V(C^{(p)}) = R(g) \cdot V(C^{(p)}).$$
- For some $V(C^{(p)})$, R is a faithful rep. //

Higher-form symmetry is abelian (at least when X has a trivial topology).

(Proof).

Since U_g has a large codimensi $p+1 > 1$,
$$U_{g_1}(M_{d-p-1}^{+\varepsilon})\, U_{g_2}(M_{d-p-1}^{-\varepsilon}) = U_{g_2}(M_{d-p-1}^{+\varepsilon})\, U_{g_1}(M_{d-p-1}^{-\varepsilon}).$$



Taking $\varepsilon \to 0$, we get
$$U_{g_1 g_2} = U_{g_2 g_1}.$$
//

Definiti (SSB of p-form symmetry)

If some $V(C^{(p)})$ with nontrivial rep. R show
$$V(C^{(p)}) \not\to 0 \quad \text{as} \quad C^{(p)} \to \infty,$$
then we say that p-form symmetry is spontaneously broken. //

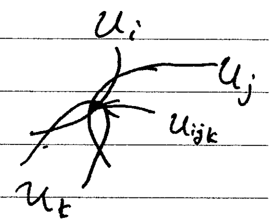
Example 4d Maxwell theory has $U(1) \times U(1)$ 1-form sym. This is spontaneously broken, and photons are NG boson for this SSB.

(Quiz : Check this example.)

Now, let us check that $SU(N)$ YM has $\mathbb{Z}_N$ 1-form symmetry,

$$W(C) \longmapsto e^{\frac{2\pi i}{N}} W(C). \qquad \text{(center symmetry)}$$

We now take a sufficiently fine open cover $\{\mathcal{U}_i\}$. Then, triple overlap $\mathcal{U}_{ijk}$ can be regarded as a codim. 2 object.



We define $\mho_{e^{2\pi i/N}}(M_2)$ as follows:
If $\mathcal{U}_{ijk} \cap M_2 \neq \emptyset$, we change the cocycle condition from

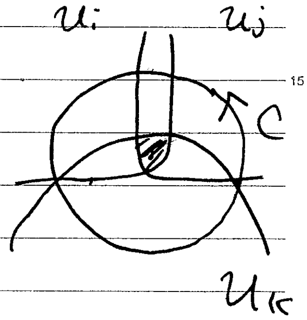
$$g_{ki}\, g_{jk}\, g_{ij} = 1$$

to

$$g_{ki}\, g_{jk}\, g_{ij} = e^{2\pi i/N}.$$

Then, If $W(C)$ links M_2, $W \mapsto e^{2\pi i/N} W$;

$$\begin{aligned} W(C) &= \mathrm{tr}\left[g_{ki} (P e^{i\int a_k}) g_{jk} (P e^{i\int a_j}) g_{ij} (P e^{i\int a_i}) \right] \\ &\mapsto e^{\frac{2\pi i}{N}} \mathrm{tr}\left[g_{ki} (P e^{i\int a_k}) g_{jk} (P e^{i\int a_j}) g_{ij} (P e^{i\int a_i}) \right] \\ &= e^{\frac{2\pi i}{N}} W(C). \end{aligned}$$



If C does not link M_2, then $W(C) \mapsto W(C)$.

This also explains that $\mho(M_2)$ is topological.
Since $\mathrm{tr}[G \wedge * G]$ is nothing but $\mathrm{Re}(W(C))$ for infinitesimally small C, $\mathrm{tr}[G \wedge * G]$ does not change. Thus, the path-integral weight is inv. under the insertion of $\mho(M_2)$ by renormalization. //

From now on, we assume that the $SU(N)$ YM is confining, i.e. Clay's problem is positive.

- Large-N analysis of θ dependence

We take the limit $N \to \infty$ by keeping $\lambda_t = g^2 N$ fixed.
Then

$$S_{YM} = \frac{1}{2g^2}\int \mathrm{tr}[G\wedge * G] + i\frac{\theta}{8\pi^2}\int \mathrm{tr}[G\wedge G]$$
$$= N\left\{ \frac{1}{2\lambda_t}\int \mathrm{tr}[G\wedge * G] + i\frac{\theta/N}{8\pi^2}\int \mathrm{tr}[G\wedge G]\right\}.$$

The ground-state energy is given by

$$e^{-E(\theta)} = \int \mathcal{D}a\, e^{-N\left\{\frac{1}{2\lambda_t}\int \mathrm{tr}[G\wedge *G] + i\frac{\theta/N}{8\pi^2}\int \mathrm{tr}[G\wedge G]\right\}}.$$

Since there are N^2 gluons, the energy should have the following form

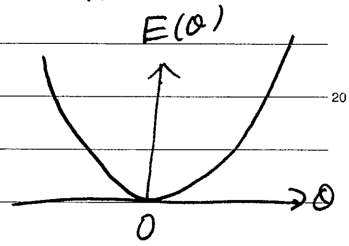
$$E(\theta) = N^2 f\left(\frac{\theta}{N}\right),$$

where f is an N-dependent universal function.
Since CP (or T) flip $\theta \mapsto -\theta$, f should be an even function;

$$f(x) = f(-x)$$

When $\theta \sim \mathcal{O}(1)$, the Taylor expansion should be applicable, and we get

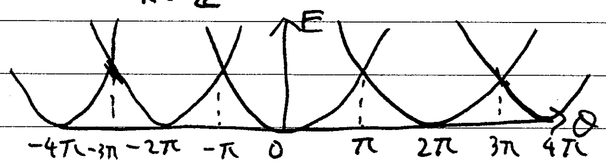
$$E(\theta) = \frac{\chi}{2}\theta^2 + \cancel{\mathcal{O}\left(\frac{\theta^4}{N^2}\right)}.$$



χ is called the topological susceptibility.
(Reflection positivity says $\chi \geq 0$).

However, this answer is inconsistent with the periodicity $\theta \sim \theta + 2\pi$. Witten '80, '98 introduces the multi-branch structure to make the story consistent:

$$E(\theta) = \min_{k\in\mathbb{Z}} N^2 f\left(\frac{\theta+2\pi k}{N}\right) = \min_{k\in\mathbb{Z}} \frac{\chi(\theta+2\pi k)^2}{2}.$$



At $\theta = \pi$, there are two vacua, which break CP spontaneously. This is called Dashen phenomenon.

• $\mathcal{N}=1$ Super Yang-Mills

We add one adjoint Weyl fermion (gluino) to $SU(N)$ YM, then the theory acquires $\mathcal{N}=1$ supersymmetry.

$$\frac{1}{2g^2}\,\mathrm{tr}\left[\bar{\tilde{g}}_R\,\bar{\sigma}^\mu\,(\partial_\mu + i[a_\mu,\cdot\,])\,\tilde{g}_L\right]$$

The classical Lagrangian is inv. under $U(1)_R$,

$$\tilde{g} \mapsto e^{i\alpha}\tilde{g}\ ,\quad \bar{\tilde{g}} \mapsto \bar{\tilde{g}}\,e^{-i\alpha},$$

but the path-integral measure is not:

$$\mathcal{D}\bar{\tilde{g}}\,\mathcal{D}\tilde{g} \mapsto \mathcal{D}\bar{\tilde{g}}\,\mathcal{D}\tilde{g}\cdot \exp\left[i\alpha\cdot\frac{2N}{8\pi^2}\int \mathrm{tr}[G\wedge G]\right]$$

(This is also called anomaly. To distinguish it from 't Hooft anomaly, we call it ABJ anomaly (Adler, Bell, Jackiw))

If $\alpha \in \frac{2\pi}{N}\mathbb{Z}$, the path-integral measure is also inv., and thus $\mathcal{N}=1$ SYM has the sym. $(\mathbb{Z}_{2N})_R$ ($\subset U(1)_R$).

It is widely believed that there is a gluino condensation in $\mathcal{N}=1$ SYM:

$$\langle \mathrm{tr}[\tilde{g}\tilde{g}]\rangle_k = \Lambda^3 e^{\frac{2\pi i k}{N}} \qquad (k=0,\cdots,N-1),$$

and thus SSB occurs as

$$\mathbb{Z}_{2N} \to \mathbb{Z}_2 .$$

(This $\mathbb{Z}_2$ is fermionic parity, $\tilde{g}\mapsto -\tilde{g}$. Since $\mathbb{Z}_2 \subset Spin(4)$, the Lorentz invariant vacua cannot break $\mathbb{Z}_2$.)

Using classical $U(1)_R$, θ can be eliminated by setting $\alpha = -\frac{\theta}{2N}$.
$\Rightarrow$ $E(\theta)$ has no θ dependence.

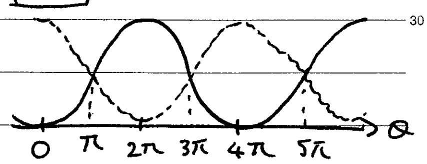
We therefore add the small mass to gluino, then

$$\mathcal{L}_{mass} = m\left(e^{i\frac{\theta}{N}}\mathrm{tr}[\tilde{g}\tilde{g}] + e^{-i\frac{\theta}{N}}\mathrm{tr}[\bar{\tilde{g}}\bar{\tilde{g}}]\right).$$

Since m is small,

$$E(\theta) = -\langle \mathcal{L}_{mass}\rangle = -m\Lambda^3\cos\left(\frac{\theta+2\pi k}{N}\right).$$

$\boxed{N=2}$



At $\theta=\pi$, CP is again spontaneously broken.

- <u>Anomaly matching</u>

In two previous analyses, we learned that the vacua at $\theta = \pi$ are nontrivial. This is explained by anomaly in Gaiotto, Kapustin, Komargodski, Seiberg, '17.

We gauge $\mathbb{Z}_N$ 1-form symmetry. It corresponds to insert 't Hooft magnetic flux $B \in H^2(X, \mathbb{Z}_N)$.
This $\mathbb{Z}_N$ two-form gauge field B can be realized as a pair of $U(1)$ 2-form and 1-form gauge fields $(B^{(2)}, B^{(1)})$, with

$$N B^{(2)} = d B^{(1)}.$$

We then introduce $U(N)$ gauge connection $\tilde{a}$ by

$$\tilde{a} = a + \frac{1}{N} B^{(1)} \mathbb{1}_N .$$

(More correctly speaking, we perform the path integral $\int \mathcal{D}\tilde{a}$ for $U(N)$ gauge field satisfying $\mathrm{tr}(\tilde{a}) = B^{(1)}$).
We define the gauged action by minimal coupling so that it is invariant under $U(1)$ 1-form gauge transformation

$$\begin{cases} B^{(2)} \mapsto B^{(2)} + d\lambda, \\ B^{(1)} \mapsto B^{(1)} + N\lambda, \\ \tilde{a} \mapsto \tilde{a} + \lambda . \end{cases}$$

Then, we get $(\tilde{G} = d\tilde{a} + i\tilde{a}^2)$

$$S = \frac{1}{2g^2} \int \mathrm{tr}\left[(\tilde{G} - B^{(2)}) \wedge * (\tilde{G} - B^{(2)})\right] + i\frac{\theta}{8\pi^2} \int \mathrm{tr}\left[(\tilde{G} - B^{(2)}) \wedge (\tilde{G} - B^{(2)})\right]$$

Importantly, the topological charge is no longer integers:

$$\begin{aligned} Q &= \frac{1}{8\pi^2} \int \mathrm{tr}\left[(\tilde{G} - B^{(2)}) \wedge (\tilde{G} - B^{(2)})\right] \\ &= \frac{1}{8\pi^2} \int \left\{ \mathrm{tr}[\tilde{G} \wedge \tilde{G}] - 2 \underbrace{\mathrm{tr}[\tilde{G}]}_{= B^{(2)} \cdot N} \wedge B^{(2)} + B^{(2)} \wedge B^{(2)} \underbrace{\mathrm{tr}(1)}_{= N} \right\} \\ &= \underbrace{\frac{1}{8\pi^2} \int \mathrm{tr}[\tilde{G} \wedge \tilde{G}]}_{\in \mathbb{Z} \text{ by } U(N) \text{ index thm}} - \underbrace{\frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)}}_{\in \frac{1}{N}\mathbb{Z} \text{ by } U(1) \text{ index th'm.}} \end{aligned}$$

Let us apply the CP transformation at $\theta = \pi$, then

$$
\begin{aligned}
S &= \frac{1}{2g^2}\int \mathrm{tr}\left[|(\tilde{G} - B^{(1)})|^2\right] + i\frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\tilde{G}-B^{(2)})^2\right] \\
&\mapsto \frac{1}{2g^2}\int \mathrm{tr}\left[|(\tilde{G} - B^{(1)})|^2\right] - i\frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\tilde{G}-B^{(2)})^2\right] \\
&= S - i\frac{2\theta}{8\pi^2}\int \mathrm{tr}\left[(\tilde{G}-B^{(2)})^2\right] \\
&= S - i\frac{2\pi}{8\pi^2}\int \mathrm{tr}\left[(\tilde{G}-B^{(2)})^2\right].
\end{aligned}
$$

We therefore obtain that

$$
e^{-S} \xmapsto{CP} e^{-S}\, e^{2\pi i\left(\cancel{\frac{1}{8\pi^2}\int \mathrm{tr}[\tilde{G}^2]} - \frac{N}{8\pi^2}\int B^{(2)2}\right)}
$$

and then

$$
Z_{\theta=\pi}[B^{(2)}, B^{(1)}] \mapsto Z_{\theta=\pi}[B^{(2)}, B^{(1)}]\, e^{-i\frac{N}{4\pi}\int B^{(2)}\wedge B^{(2)}}.
$$

We should check whether this is the genuine anomaly.
Possible 4d local counter term is

$$
i\frac{Nk}{4\pi}\int B^{(2)}\wedge B^{(2)},
$$

with $k \in \mathbb{Z}_N$:

$$
Z_{\theta=\pi, k} \equiv Z_{\theta=\pi}\cdot e^{i\frac{Nk}{4\pi}\int B^{(2)}\wedge B^{(2)}}.
$$

$$
Z_{\theta=\pi, k} \xmapsto{CP} Z_{\theta=\pi, k}\, e^{-i\frac{N(2k+1)}{4\pi}\int B^{(2)}\wedge B^{(2)}}.
$$

The anomaly is absent iff $2k+1 = 0 \mod N$.

$N \in 2\mathbb{Z}$: No counter term can cancel the anomaly.
$\therefore$ There is $\mathbb{Z}_N^{[1]} \times CP$ anomaly. Anomaly matching says the low-energy physics at $\theta = \pi$ is

- CFT
- SSB of CP ($\Leftarrow$ preferred in large-N limit)
- Topological order (SSB of $\mathbb{Z}_N$ 1-form sym.)

$N \in 2\mathbb{Z}+1$: $k = \frac{N-1}{2}$ gives the counter term. In this case, there is no anomaly.
However, since k can only take discrete values, it cannot change if we change θ adiabatically from 0 to π,

at $\theta = 0$, we assume confinement with unique ground state. To cancel the fake anomaly at $\theta = 0$, we should choose $k = 0$, which is different from $k = \frac{N-1}{2}$ at $\theta = \pi$.

$\Rightarrow$ CP symmetric quasi-vacqua at $\theta = 0$ and π cannot be connected continuously so long as $\mathbb{Z}_N^{[1]}$ is respected.

(Global inconsistency condition; Gaiotto, Kapustin, Komargodski, Seiberg '17
The matching condition is examined by Tanizaki, Kikuchi, '17)

The possible IR physics at $\theta = \pi$ for odd N is

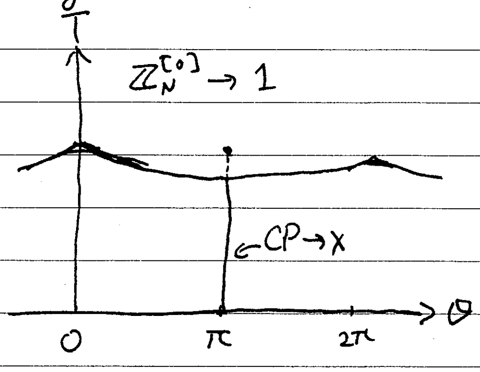
- CFT,
- SSB of CP (← Large-N prefers.)
- Topological order, or
- $\theta = 0$ and π are distinct as SPT protected by $\mathbb{Z}_N^{[1]}$

(Quiz: Read 't Hooft '81 "Confinement phenomenon in..." § XV., and discuss how that scenario matches the anomaly)

We can also discuss the phase diagram of YM at finite T and θ using anomaly. By compactifying the imaginary-time direction as S^1, the effective 3d theory has the symmetry:

$$((\mathbb{Z}_N \text{ 0-form}) \times (\mathbb{Z}_N \text{ 1-form})) \rtimes CP \text{ at } \theta = 0 \text{ or } \pi.$$

As a consequence of 4d anomaly for $\mathbb{Z}_N^{[1]} \rtimes CP$, 3d theory also has the anomaly for $(\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[1]}) \rtimes CP$ at $\theta = \pi$.



Assuming $\mathbb{Z}_N^{[1]}$ is unbroken, anomaly matching says $\mathbb{Z}_N^{[0]}$ or CP should be broken at $\theta = \pi$.
A possible phase diagram is given in above figure.

4-2. QCD, chiral symmetry breaking, and anomaly

Next we want to discuss 4d quantum chromodynamics (QCD), which is a model of strong interaction of the Standard Model. The theory is defined by

$$S = \frac{1}{2g^2}\int \mathrm{tr}[G\wedge * G] + \int \sum_{f=1}^{N_f} \left(\overline{\psi}_f \gamma^\mu (\partial_\mu + i a_\mu)\psi_f + m_f \overline{\psi}_f \psi_f\right) d^4x .$$

a : $SU(N_c)$ gauge field with $G = da + i a^2$.

We consider the special case $m_f = 0$, which is called chiral limit. In this limit, the theory acquires the chiral symmetry

$$G = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}} .$$

Standard lore for the low-energy physics

- $N_f^* < N_f < \frac{11}{2} N_c$ Conformal behavior
- $N_f < N_f^*$ Chiral symmetry breaking $G \to H = \frac{SU(N_f)_V \times U(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}}$.

(When $N_f > \frac{11}{2} N_c$, QCD is IR-free).

Caswell - Banks - Zaks fixed point

The perturbative β function $\beta(g) = \frac{dg}{d\ln\mu}$ is given as

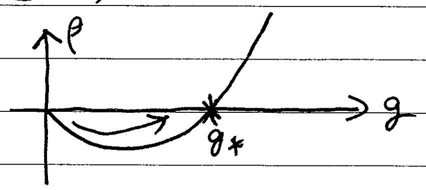
$$\beta(g) = -b_0 g^3 + b_1 g^5 + \cdots$$

with

$$b_0 = \frac{1}{16\pi^2}\frac{11N_c - 2N_f}{3}, \quad b_1 = -\frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \frac{1}{2}N_f\left(2\frac{N_c^2-1}{N_c} + \frac{20}{3}N_c\right)\right).$$

If $N_f \lesssim \frac{11}{2} N_c$, then $-b_0 < 0$ and $b_1 > 0$.

(Indeed, $b_1 > 0$ if $N_f > \frac{34 N_c^3}{(13N_c^2 - 3)}$).

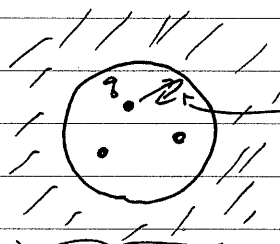


Then, RG flow stops at $g = g_*$, and interacting CFT will appear.

Chiral symmetry breaking

If N_f is not so large, perturbative β function does not have zeros, and $g(\mu = \Lambda) = \infty$. This suggests the appearance of infrared energy scale Λ dynamically.

Indeed, in strong-interaction sector of our world, the asymptotic states are given by color-singlet objects, hadrons, of size $\sim \Lambda^{-1}$. Assume that this feature persists for $m_f = 0$.



pion cloud

chirality flips since helicity does

$\Downarrow$

Outside hadron, QCD vacuum should break chiral symmetry.

$\langle \bar{\Psi} \Psi \rangle \sim \Lambda^3$, and

$$SU(N_f)_L \times SU(N_f)_R \longrightarrow SU(N_f)_V.$$

The effective theory is described by NG bosons, which is nonlinear sigma model with the target space

$$\frac{G}{H} \simeq \frac{SU(N_f)_L \times SU(N_f)_R}{SU(N_f)_V} \simeq SU(N_f).$$

In both phases, low-energy limit of massless QCD is not trivial. This can be understood as a consequence of 't Hooft anomaly matching.

Perturbative chiral anomaly and anomaly matching

QCD with massless quarks has chiral symmetry $SU(N_f)_L \times SU(N_f)_R$:
$\psi_L \equiv P_L \psi$, $\psi_R \equiv P_R \psi$, and

$$\begin{cases} U_L \ni SU(N_f)_L & \psi_L \mapsto U_L \psi_L \\ U_R \ni SU(N_f)_R & \psi_R \mapsto U_R \psi_R \end{cases}$$

$$\left(\bar\psi \gamma^\mu D_\mu \psi = \bar\psi_L \not{D} \psi_L + \bar\psi_R \not{D} \psi_R, \quad m\bar\psi\psi = m(\bar\psi_L \psi_R + \bar\psi_R \psi_L) \right)$$

We introduce the background gauge field L, R for $SU(N_f)_L$, $SU(N_f)_R$, resp.
Using the result of Sec. 2, we find that

$$Z_{QCD}[L,R]\, e^{i N_c (CS_5(L) - CS_5(R))}$$

is gauge inv., where

$$CS_5[A] = \int_{M_5} \frac{1}{24\pi^2} \mathrm{tr}\left[AF^2 - \frac{1}{4} A(FA + AF) + \frac{1}{10} A^5 \right]$$

To match the anomaly, we need
$\begin{cases} \cdot \text{ CFT, or} \\ \cdot \text{ chiral symmetry breaking.} \end{cases}$

cf. In the case of Super QCD, (Seiberg '94)

- $N_f < N_c$: runaway vacuum
- $N_f = N_c$: chiral symmetry breaking
- $N_f = N_c + 1$: S-Confinement. (Confinement without chiral SB. There are massless fermionic hadrons)
- $N_c + 1 < N_f < \frac{3}{2} N_c$: Magnetic dual is IR-free.
- $\frac{3}{2} N_c < N_f < 3N_c$: IR fixed point (CFT)
- $N_f > 3N_c$: IR-free.

This realizes the situation where 't Hooft originally wanted to find by anomaly matching.

In the chiral broken phase, we need to add the Wess-Zumino term in order to match the anomaly : $\int_{M_5} \frac{1}{240\pi^2} \mathrm{tr}[(U^\dagger dU)^5]$.

$\rightsquigarrow$ $\pi^0 \to 2\gamma$, $KK \to \pi\pi\pi$, ...

Discrete anomaly of massless QCD

(Tanizaki '18, See also Shimizu, Yonekura '17, Tanizaki, Misumi, Sakai '17, Gaiotto, Komargodski, Seiberg '17)

Now, let us pay more attenti to the global structure of the symmetry group:

$$G = \frac{SU(N_f)_L \times SU(N_f)_R \times U(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}}.$$

To enlighten the global structure, we pay attenti to the vector-like symmetry and also "discrete chiral symmetry":

$$G \supset \underbrace{\frac{SU(N_f)_V \times U(1)_V}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}}}_{H} \times (\mathbb{Z}_{N_f})_L .$$

A motivati to consider this subgroup comes out of an exotic scenario about chiral symmetry breaking. (Stern, '97, '98).
Usually, we consider that SSB occurs as

$$\langle \bar{\psi}_L \psi_R \rangle \neq 0$$

and then

$$G \xrightarrow{SSB} H.$$

We can however propose a different order parameter

$$\langle (\bar{\psi}_R T^a \psi_L) \cdot (\bar{\psi}_L T^a \psi_R) \rangle \neq 0 \quad \text{w/ } \langle \bar{\psi}\psi \rangle = 0,$$

and then

$$G \xrightarrow{SSB} H \times (\mathbb{Z}_{N_f})_L .$$

(cf. This possibility was excluded from zero-density QCD wo. scalar fields by QCD inequalities (Kogan, Kovner, Shifman '98).
Until recently, it was still open if this can also be excluded at finite μ_B)

Let's discuss if this exotic possibility is allowed by anomaly.
However, the local str. of the target space of phenomenological Lagrangian is the same with the orthodox one, perturbative anomaly matching is insufficient for this purpose.

We introduce the background gauge field for $H \times (\mathbb{Z}_{N_f})_L$:

vector-like part H:

- $SU(N_f)$ 1-form gauge field : A_f
- $U(1)$ 1-form gauge field : A_B
- $\mathbb{Z}_{N_c}$ 2-form gauge field : B_c ('t Hooft magnetic flux for color gauge field)
- $\mathbb{Z}_{N_f}$ 2-form gauge field : B_f ('t Hooft magnetic flux for flavor gauge field)

chiral part $(\mathbb{Z}_{N_f})_L$:

- $\mathbb{Z}_{N_f}$ 1-form gauge field : A_χ.

We find that Z_{QCD} requires 5d SPT

$$Z_{QCD} \cdot \exp\left(i \frac{N_f}{(2\pi)^2} \int A_\chi \wedge N_c(dA_V + B_c) \wedge B_f\right)$$

for gauge invariance. This means that the baryon number conservation is anomalously violated:

$$\partial_\mu J^\mu_B = \frac{N_f}{(2\pi)^2} dA_\chi \wedge B_f.$$

When chiral symmetry is broken in an orthodox way, $G \to H$, then

$$J_B \xrightarrow{\text{identify}} J_{\text{skyrmion}} = \frac{1}{24\pi^2} \mathrm{tr}[(U^\dagger dU)^3]. \qquad (U \in \tfrac{G}{H} \simeq SU(N_f))$$

In the existence of the above background gauge fields, it becomes

$$J_{\text{skyrmion}} = \frac{1}{24\pi^2} \mathrm{tr}[(U^\dagger D U)^3] + \frac{1}{8\pi^2} \mathrm{tr}[(UDU^\dagger)(F_f + dA_\chi) - (U^\dagger DU)F_f],$$

and then

$$dJ_{\text{skyrmion}} = \frac{N_f}{(2\pi)^2} dA_\chi \wedge B_f.$$

$\rightarrow$ Anomaly matching is satisfied for ~~of~~ discrete one, too.

In exotic case, $\tilde{U} \in \frac{G}{H \times (\mathbb{Z}_{N_f})_L} \simeq SU(N_f)/\mathbb{Z}_{N_f}$.

Doing the same computation, we can find

$$dJ_{\text{skyrmion}} = \frac{N_f}{(2\pi)^2} \underbrace{\mathrm{tr}[\tilde{U}^\dagger d\tilde{U}]}_{\text{dynamical } (\mathbb{Z}_{N_f}) \text{ 1-form gauge field}} \wedge B_f \neq dJ_{\text{skyrmion}}.$$

$\rightarrow$ Anomaly matching forbids Stern phase from QCD ground states.

(Quiz: Check the discrete anomaly matching for Seiberg dualities).